Bulletin of Engineering Geology and the Environment (2022) 81:11
https://doi.org/10.1007/s10064-021-02501-1

ORIGINAL PAPER

Elastoplastic analytical solution of circular ring expansion problem for bi-modulus material based on SMP yield criterion

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Received: 23 June 2021 / Accepted: 9 November 2021

Abstract

It is known that there are great types of materials, especially the geotechnical materials, e.g., rock, soil, with obvious differences in elastic modulus and Poisson's ratio under tension and compression in nature. However, the current investigation for this type of material is still not sufficient. In this study, the elastoplastic analytical solutions of stress and displacement for the circular ring expansion problem are derived based on the SMP yield criterion and bi-modulus theory. The effect of the bi-modulus characteristics of materials on the distribution of stress, strain, and displacement, and the effect of friction angle on the critical internal pressure P_c and the ultimate internal pressure P_u in the expansive circular ring are then further investigated. It is shown by the analytical results that the distribution pattern and the magnitude of stress, strain, and displacement in the expansive circular ring, as well as the critical internal pressure P_c, the ultimate internal pressure P_u applied on the internal radius of the circular ring are all significantly affected by the modulus ratio, i.e., $R = E_t/E_c = \nu_t/\nu_c$. ($E_c$ and E_t are respectively the elastic modulus under compression and tension;ν_c and ν_t are respectively the Poisson ratios under compression and tension). Based on the proposed analytical solution, the effect of the SMP yield criterion and Mohr-Coulomb yield criterion on the stress in the plastic zone in the circular ring is compared. It is found that the stress in the plastic zone in the circular ring is overestimated by the Mohr-Coulomb yield criterion.

Keywords Circular ring expansion problem · Bi-modulus theory · SMP yield criterion · Mohr-Coulomb yield criterion · Elastoplastic analytical solution · Tensile elastic modulus

Introduction

Circular ring expansion theory has been widely used in the practice of engineering; for example, the pressuremeter test (Clarke 1994; Yu and Mitchell 1999) was widely adopted to measure the deformation modulus of foundation, and the bearing capacity of foundation could be further estimated. According to the circular ring expansion theory, the stability and deformation of tunnels were evaluated by Teraghi and Richart (1952) and Hoek and Brown (1980), and the bearing capacity of piles was estimated by Randolph et al. (1979). Therefore, the study on the circular ring expansion problem for geotechnical medium has a great practical value for the optimum design in civil engineering.

The circular ring expansion theory was first proposed by Bishop et al. (1945) and used to study the metal indentation problem. It has then been applied in the field of geotechnical engineering by Menard (1957) and Gibson and Anderson (1961). Vesic (1972) had given a general solution to the circular ring expansion problem and extended it to many other fields, such as the pile foundation design. Besides, Yu (2000) also has made a systematic contribution in this field.

Currently, the elastic analytical solution for the circular ring expansion problem can be easily found in literature, such as Timoshenko et al. (1970). However, there are only a few elastoplastic analytical solutions for this problem, due to the fact it is difficult to select a suitable yield criterion that is applicable and easy for theoretical derivation. For example, for metal materials, most predecessors established the elastoplastic solutions based on the Tresca yield criterion (Hill 1950; Xu and Liu 1995). For geotechnical materials, the elastoplastic analytical solutions were mostly based on the

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Published online: 11 December 2021

Mohr-Coulomb (Florence and Schwer 1978) or Hoek-Brown yield criterion (Brown et al. 1983). Besides, based on the unified strength theory, Wang et al. (2019) derived a formulation of stress in a double-layered think-walled cylinder with uniform internal pressure by considering the bi-modulus of material. Zhu et al. (2020) proposed a unified elastoplastic solution for a double-layered or multilayered cylinder by introducing the ratio of the tensile strength σ_t to the compressive strength σ_c. In addition, Gao et al. (2017) established a modified semi-analytical model to study the mechanical response of a bi-modulus cylinder placed in a symmetrical temperature field. However, since the influence of intermediate principal stress σ_2 was not considered in most of the above yield criteria, it brought great convenience for the derivation of a plastic analytical solution. However, the reliable demonstrations by Single et al. (1998) and Sãyao and Vaid (2011) showed that the intermediate principal stress σ_2 had a significant influence on the strength of materials. Therefore, it is of practical value to establish the elastoplastic analytical solution for the circular ring expansion problem based on the yield criterion that considers the effect of intermediate principal stress σ_2.

Most importantly, for the problem of circular ring expansion, materials were all treated as an ideal elastoplastic material with the same elastic modulus and Poisson's ratio under tension and compression in most previous works. However, the elastic modulus and Poisson's ratio of many natural materials are markedly different under tension and compression, such as concrete, ceramics, rock, and other materials (Ye et al. 2009, 2012; Zhang et al. 2018) in our natural world. If this kind of property of materials is ignored, and the classical elastic theory is still used for determining the deformation of these materials, a considerable error would be caused. For example, Sundaram and Corrales (1980) found that, when the modulus ratio, i.e., E_c/E_t, is equal to 10, the error on the deformation between the results obtained by considering the modulus difference of materials under tension and compression, and the result predicted by the classical solution could reach 40% (E_c and E_t are respectively the elastic modulus of materials under compression and tension). Therefore, it is essential to consider the property of bi-modulus materials in the analytical solution for the circular ring expansion problem.

In view of the shortcomings in the previous works, this study establishes an analytical solution for the circular ring expansion of ideal elastoplastic materials based on the bi-modulus theory proposed by Ambartsumyan (1986) by introducing the SMP criterion to consider the effect of intermediate principal stress (Matsuoka and Nakai 1974). There are three reasons for selecting the SMP yield criterion in this study. Firstly, the SMP yield criterion is evolved from the Mohr-Coulomb yield criterion. In the π plane, the yield surfaces of the two yield criteria are very close; especially at the six corners, the SMP yield surface is circumscribed with the Mohr-Coulomb yield surface, as illustrated in Fig. 1b. It is shown that the applicability of the SMP yield criterion is as good as the Mohr-Coulomb yield criterion. Secondly, the effect of intermediate principal stress is considered in the SMP criterion. Thirdly, the expression form of the SMP yield criterion is convenient for the mathematical derivation due to the smoothness of its yield surfaces in the π plane. In summary, due to the consideration of the effect of the intermediate principal stress, and the bi-modulus property of materials, the analytical solution proposed in this study will be more practical than the previous analytical solutions in engineering.

Yield criterion

The SMP is an excellent yield criterion with a simple form and clear physical meaning. The expression of this criterion in 3D principal stress space and plane strain is introduced in the following.

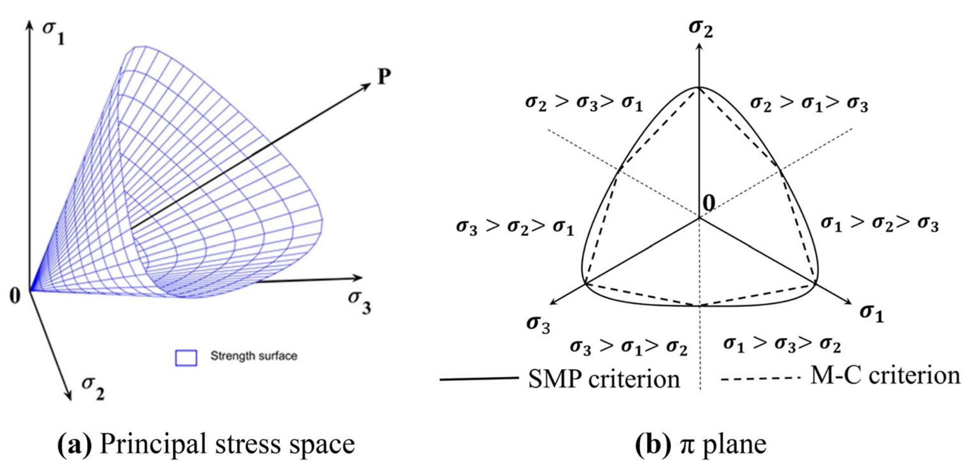


Fig. 1 Geometry of SMP yield surface in the principal stress space and on the π plane. **a** Principal stress space. **b** π plane

SMP yield criterion in 3D principal stress space

In the Mohr-Coulomb yield criterion, when the ratio of shear stress to normal stress on a shearing plane satisfies the following equation, the material will yield, as illustrated in Fig. 2a.

$$\frac{\tau}{\sigma_n}=\frac{\sigma_i-\sigma_j}{2\sqrt{\sigma_i\sigma_j}}=\frac{(\sigma_i-\sigma_j)/2}{\sqrt{((\sigma_i+\sigma_j)/2)^2+((\sigma_i-\sigma_j)/2)^2}}=\tan\varphi_{ij} \tag{1}$$

where τ and σ_n are the shear and normal stress on the shearing plane, respectively; φ_{ij} represents the friction angle of materials. σ_i, σ_j are the maximum and minimum principal stress when yielding.

Based on the Mohr-Coulomb yield criterion, Matsuoka and Nakai (1974) proposed the SMP yield criterion, in which the influence of the intermediate principal stress σ_2 was considered. Materials will be judged as being yield state by the SMP yield criterion when the ratio of shear stress to normal stress on a spatial shearing plane reaching a certain value (Luo et al. 2010), and satisfying the following equation:

$$\frac{\tau_{\mathrm{SMP}}}{\sigma_{\mathrm{SMP}}}=\frac{2}{3}\sqrt{\left(\tan\varphi_{12}\right)^2+\left(\tan\varphi_{23}\right)^2+\left(\tan\varphi_{13}\right)^2}$$

$$=\frac{2}{3}\sqrt{\left(\frac{\tau_{P3}}{\sigma_{P3}}\right)^2+\left(\frac{\tau_{P2}}{\sigma_{P2}}\right)^2+\left(\frac{\tau_{P1}}{\sigma_{P1}}\right)^2}$$

$$=\frac{2}{3}\sqrt{\left(\frac{\sigma_1-\sigma_2}{2\sqrt{\sigma_1\sigma_2}}\right)^2+\left(\frac{\sigma_2-\sigma_3}{2\sqrt{\sigma_2\sigma_3}}\right)^2+\left(\frac{\sigma_1-\sigma_3}{2\sqrt{\sigma_1\sigma_3}}\right)^2}$$

$$=\sqrt{\frac{I_1I_2-9I_3}{9I_3}}=\text{Constant} \tag{2}$$

where τ_{SMP} and σ_{SMP} are the shear stress and normal stress on the spatial shearing plane when materials reaching yield state by the SMP yield criterion, respectively; σ_1, σ_2, σ_3 are the principal stresses when yielding.

Under the situation of triaxial compression ($\sigma_1>\sigma_2=\sigma_3$), Eq. (2) can be simplified as follows:

$$\frac{\tau_{\mathrm{SMP}}}{\sigma_{\mathrm{SMP}}}=\frac{\sqrt{2}}{3}\frac{\sigma_1-\sigma_3}{\sqrt{\sigma_1\sigma_3}}=\text{Constant} \tag{3}$$

Since a Mohr-Coulomb material meets the following equation in the triaxial compression test:

$$\frac{\tau}{\sigma_n}=\tan\varphi=\frac{\sigma_1-\sigma_3}{2\sqrt{\sigma_1\sigma_3}} \tag{4}$$

Under the situation of triaxial compression, the Mohr-Coulomb yield surface and the SMP yield surface coincide at a vertex on the π plane. As a consequence, τ_{SMP}/σ_{SMP} in Eq. (3) is equal to τ/σ_n in Eq. (4) under this situation. Substituting Eqs. (3) and (4) into Eq. (2), the relationship between $\frac{\tau_{\mathrm{SMP}}}{\sigma_{\mathrm{SMP}}}$ and the friction angle φ can be obtained.

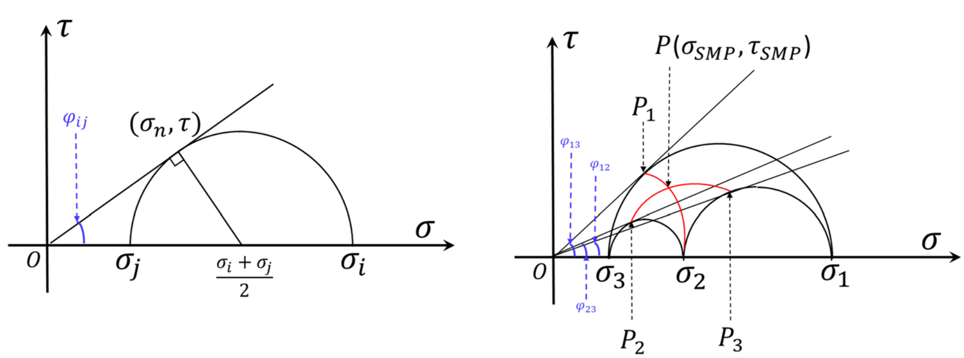
$$\frac{\tau_{\mathrm{SMP}}}{\sigma_{\mathrm{SMP}}}=\sqrt{\frac{I_1I_2-9I_3}{9I_3}}=\frac{2\sqrt{2}}{3}\tan\varphi \tag{5}$$

where φ represents the internal friction angle of materials that can be tested by the triaxial compression test; I_1, I_2, and I_3 are the stress invariants when yielding, and their expressions are as follows:

$$\begin{cases} I_1=\sigma_1+\sigma_2+\sigma_3 \\ I_2=\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_1\sigma_3 \\ I_3=\sigma_1\sigma_2\sigma_3 \end{cases} \tag{6}$$

The general expression of the SMP yield criterion can be obtained by simplifying Eq. (5):

$$F\left(I_1,I_2,I_3\right)=\frac{I_1I_2}{I_3}-8\tan^2\varphi-9=0 \tag{7}$$



(a) Mohr-Coulomb yield criterion **(b)** SMP yield criterion

Fig. 2 Shear stress and normal stress on the shearing plane when yielding. **a** Mohr-Coulomb yield criterion. **b** SMP yield criterion

Because the yield criterion given by Eq. (7) is only applicable to non-cohesive granular materials, i.e., the materials without cohesion, i.e., $C = 0$. To overcome this shortcoming, Matsuoka and Nakai (1974) further proposed that the yield surface moves a distance $\sigma_0 = C\cot\varphi$, to the negative axis direction along the principal stress axis σ, as demonstrated in Fig. 3. After this moving, a new coordinates $\hat{\sigma} - \hat{0} - \hat{\tau}$ can be established. The relationship between $\hat{\sigma} - \hat{0} - \hat{\tau}$ and $\sigma - 0 - \tau$ can be expressed by Eq. (8).

$$\begin{cases} \hat{\sigma} = \sigma + \sigma_0 \\ \hat{\tau} = \tau \end{cases} \tag{8}$$

where σ_0 is the moving distance of the SMP yield surface, which is related to the cohesion of material, and its expression is

$$\sigma_0 = \frac{C}{\tan\varphi} = C\cot\varphi \tag{9}$$

By substituting Eq. (8) into Eq. (7), the extended SMP yield criterion can be formulated as

$$F\left(\hat{I}_1, \hat{I}_2, \hat{I}_3\right) = \frac{\hat{I}_1\hat{I}_2}{\hat{I}_3} - 8\tan^2\varphi - 9 = 0 \tag{10}$$

where the expressions of $\hat{I}_1, \hat{I}_2$, and $\hat{I}_3$ are as follows:

$$\begin{cases} \hat{I}_1 = \hat{\sigma}_1 + \hat{\sigma}_2 + \hat{\sigma}_3 = I_1 + 3\sigma_0 \\ \hat{I}_2 = \hat{\sigma}_1\hat{\sigma}_2 + \hat{\sigma}_2\hat{\sigma}_3 + \hat{\sigma}_3\hat{\sigma}_1 = I_2 + 2I_1\sigma_0 + 3{\sigma_0}^2 \\ \hat{I}_3 = \hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 = I_3 + I_2\sigma_0 + I_1{\sigma_0}^2 + {\sigma_0}^3 \end{cases} \tag{11}$$

SMP strength formula under plane strain

Since the circular ring expansion problem is a typical plane strain problem, it is necessary to find the expression of the SMP yield criterion under the plane strain condition. Satake (1976) proposed the stress expression for the failure of soil under the strain condition according to the associated flow law and SMP strength criterion; the expression was given as follows:

$$\sigma_2 = \sqrt{\sigma_1\sigma_3} \tag{12}$$

Substituting Eq. (12) into Eq. (6), the stress invariants expressed only by σ_1, σ_3 can be obtained:

$$\begin{cases} I_1 = \sigma_1 + \sqrt{\sigma_1\sigma_3} + \sigma_3 \\ I_2 = \sigma_1\sqrt{\sigma_1\sigma_3} + \sqrt{\sigma_1\sigma_3}\sigma_3 + \sigma_1\sigma_3 \\ I_3 = \sigma_1\sqrt{\sigma_1\sigma_3}\sigma_3 \end{cases} \tag{13}$$

Substituting Eq. (13) into Eq. (7), the expression of the SMP yield criterion under plane strain is obtained as follows:

$$\frac{\sigma_1}{\sigma_3} = K = \frac{1}{4}\left(\sqrt{8\tan^2\varphi + 9} + \sqrt{8\tan^2\varphi + 6 - 2\sqrt{8\tan^2\varphi + 9}} - 1\right)^2 \tag{14}$$

For the materials with $C \neq 0$, the principal stresses must be translated according to Eqs. (8), (9) and (14) must be modified as follows:

$$\frac{\sigma_1 + \sigma_0}{\sigma_3 + \sigma_0} = K \tag{15}$$

where K is a constant related to the friction angle of materials.

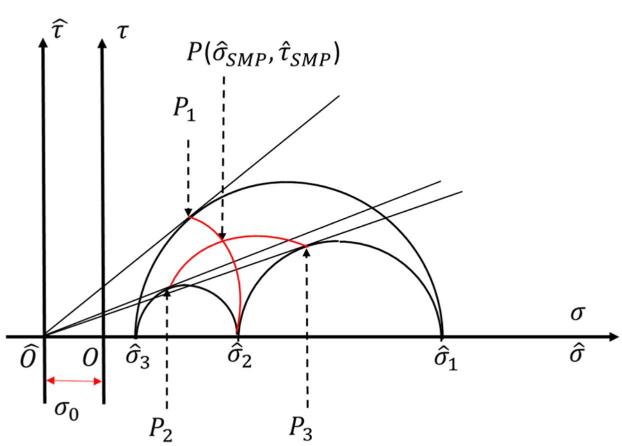


Fig. 3 Extended SMP yield criterion for cohesive materials ($C \neq 0$)

Description of the circular ring expansion problem

Under the condition of plane strain, when the ratio of external radius $r = b$ to the internal radius $r = a$ of the ring is greater than 1.2, it can be called a thick-wall ring, as shown in Fig. 4. The ring is centrosymmetric, and its thickness is constant. If the load distribution on the internal circular is also symmetrical to the central axis and the magnitude of the load is uniform, it can be treated as a plane strain axisymmetric problem. For this kind of problem, all components can be expressed in a polar coordinate system (r, θ). Moreover, the stress, strain, and displacement are only the function of the radius r, i.e.,$\sigma_r(r)$,$\sigma_\theta(r)$,$\varepsilon_r(r)$,$\varepsilon_\theta(r)$. Meanwhile, there is no shear stress at any point in the ring due to the symmetric characteristic of the circular ring, i.e.,$\tau_{r\theta} = 0$. Additionally, due to the fact that only radial uniform expansion or contraction appears in the ring, there is only radial displacement, i.e.,$u_\theta(r) = 0$. It is noticed that the displacement in the z-direction is inexistent under plane strain condition, i.e., $u_z(r) = 0$ is always satisfied.

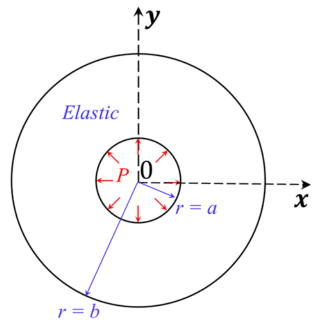


Fig. 4 Illustration of the axisymmetric circular ring expansion problem under plane strain

Basic equations

The unknown variables in the axisymmetric plane strain problems, i.e., σ_r, σ_θ, ε_r, ε_θ, u_r, must satisfy the following basic equations (Yu 2000).

1. The differential equations for stress equilibrium are as follows:

$$\begin{cases} \frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \\ \tau_{r\theta} = 0 \end{cases} \tag{16}$$

where r is the radial distance to the center; σ_r, σ_θ and σ_z are the radial, circumferential, and axial stress, respectively.

2. The geometric equations are as follows:

$$\begin{cases} \varepsilon_r = \frac{du_r}{dr} \\ \varepsilon_\theta = \frac{u_r}{r} \end{cases} \tag{17}$$

where u_r is the radial displacement.

3. The compatibility equation of deformation is:

$$r\frac{d\varepsilon_\theta}{dr} + \varepsilon_\theta - \varepsilon_r = 0 \tag{18}$$

According to the generalized elastic law, the relationship between stress and strain can be expressed as follows:

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \sigma_r \\ \sigma_\theta \end{bmatrix} \tag{19}$$

The matrix form of Eq. (19) is expressed as follows:

$$\{\boldsymbol{\varepsilon}\} = [\boldsymbol{C}]\{\boldsymbol{\sigma}\} \tag{20}$$

$$[\boldsymbol{C}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{21}$$

where $[\boldsymbol{C}]$ is the compliance matrix, $a_{ij} = f(E, \nu), (i, j = 1, 2)$. E, ν are the elastic modulus and Poisson's ratio of materials under compression. When the bi-modulus characteristics of materials are considered, a_{ij} should be defined by E_c, E_t, ν_c and ν_t i.e., $a_{ij} = f(E_c, E_t, \nu_c, \nu_t)$. According to the bi-modulus theory (Ambartsumyan 1986), to satisfy the requirement of symmetry in the matrix $[\boldsymbol{C}]$, the assumption of $\nu_t/E_t = \nu_c/E_c$ should be satisfied among E_c, ν_c, E_t, and ν_t. Otherwise, the elastoplastic analytic solution can't be derived.

For the axisymmetric circular ring expansion problem under plane strain shown in Fig. 4, the expression of a_{ij} can be given as (Luo et al. 2004; Wang et al. 2019):

$$\begin{cases} a_{11} = \frac{1}{E_c} - \frac{(-\nu_c/E_c)^2}{1/E_c} = \frac{(1-\nu_c\nu_c)}{E_c} \\ a_{22} = \frac{1}{E_t} - \frac{(-\nu_t/E_t)^2}{1/E_c} = \frac{(1-\nu_c\nu_t)}{E_t} \\ a_{12} = a_{21} = -\frac{\nu_t}{E_t} - \frac{(-\nu_t/E_t)^2}{1/E_c} = -\frac{\nu_t(1+\nu_c)}{E_t} \end{cases} \tag{22}$$

By substituting Eq. (22) into Eq. (19), the relationship taking into consideration of the bi-modulus characteristic of materials between stress and strain is obtained as follows:

$$\begin{cases} \varepsilon_r = \frac{(1-\nu_c\nu_c)}{E_c}\sigma_r - \frac{\nu_t(1+\nu_c)}{E_t}\sigma_\theta \\ \varepsilon_\theta = -\frac{\nu_t(1+\nu_c)}{E_t}\sigma_r + \frac{(1-\nu_c\nu_t)}{E_t}\sigma_\theta \end{cases} \tag{23}$$

Elastoplastic solution derivation

Elastic solution

As shown in Fig. 4, there is a pressure P applied on the inner wall ($r = a$) of the ring. If the ring is in the stage of elastic deformation, the stress σ_r and $|\sigma_\theta|$ in the ring will increase with the pressure P. To obtain the stress solution in the ring at the elastic deformation stage, Eq. (19) is substituted into Eq. (18). Then, a new differential equation for stress equilibrium can be established:

$$a_{21}\frac{d\sigma_r}{dr} + a_{22}\frac{d\sigma_\theta}{dr} + \frac{a_{21} - a_{11}}{r}\sigma_r + \frac{a_{22} - a_{12}}{r}\sigma_\theta = 0 \tag{24}$$

Meanwhile, according to Eq. (16), the relationship between σ_r and σ_θ can be expressed as follows:

$$\sigma_\theta = \sigma_r + r\frac{d\sigma_r}{dr} \tag{25}$$

By substituting Eq. (25) into Eq. (24), Eq. (24) can be rewritten as follows:

$$\frac{d^2\sigma_r}{dr^2} + \frac{3}{r}\frac{d\sigma_r}{dr} + \left(\frac{a_{22} - a_{11}}{a_{22}}\right)\frac{\sigma_r}{r^2} = 0 \tag{26}$$

The general solution of Eq. (26) for the radial stress is:

$$\sigma_r = Ar^{-(s+1)} + Br^{s-1} \tag{27}$$

where A and B are the integration constants, s is defined by:

$$s = \sqrt{\frac{a_{11}}{a_{22}}} = \sqrt{\frac{E_t(1 - \nu_c\nu_c)}{E_c(1 - \nu_c\nu_t)}} \tag{28}$$

where s is a parameter reflecting the effect of the bi-modulus characteristic of materials. Furthermore, the circular ring illustrated in Fig. 4 needs to satisfy the following boundary conditions:

$$\begin{cases} \sigma_r = P, r = a \\ \sigma_r = 0, r = b \end{cases} \tag{29}$$

Substituting the boundary conditions Eq. (29) into Eqs. (26) and (27), the elastic stress solution in the ring for bi-modulus materials is established as follows:

$$\begin{cases} \sigma_r = \frac{Pa^{s+1}}{a^{2s}-b^{2s}}(r^{s-1} - \frac{b^{2s}}{r^{s+1}}) \\ \sigma_\theta = \frac{Pa^{s+1}}{a^{2s}-b^{2s}}(sr^{s-1} + \frac{sb^{2s}}{r^{s+1}}) \end{cases} \tag{30}$$

Additionally, according to Eqs. (23) and (30), the elastic strain solution in the expansive ring for bi-modulus materials can be further obtained as follows:

$$\begin{cases} \varepsilon_r = -\frac{Pa^{s+1}}{(a^{2s}-b^{2s})E_c}\left\{\left[\left(1 - \nu_c\nu_c\right) - s(\nu_c + \nu_c\nu_c)\right]r^{s-1} - \left[\left(1 - \nu_c\nu_c\right) + s(\nu_c + \nu_c\nu_c)\right]\frac{b^{2s}}{r^{s+1}}\right\} \\ \varepsilon_\theta = -\frac{Pa^{s+1}}{(a^{2s}-b^{2s})E_t}\left\{\left[s\left(1 - \nu_c\nu_t\right) - (\nu_t + \nu_c\nu_t)\right]r^{s-1} + \left[s\left(1 - \nu_c\nu_t\right) - (\nu_t + \nu_c\nu_t)\right]\frac{b^{2s}}{r^{s+1}}\right\} \end{cases} \tag{31}$$

Substituting Eq. (31) into Eq. (17), the elastic radial displacement solution in the ring expansion for bi-modulus materials can be expressed as follows:

$$u_r = -\frac{Pa^{s+1}}{E_c(a^{2s} - b^{2s})}\left\{\left[\left(1 - \nu_c\nu_c\right) - s(\nu_c + \nu_c\nu_c)\right]\frac{r^s}{s} + \left[\left(1 - \nu_c\nu_c\right) + s(\nu_c + \nu_c\nu_c)\right]\frac{b^{2s}}{sr^s}\right\} \tag{32}$$

If the bi-modulus characteristic of materials is not considered, Eqs. (29) and (31) can degrade to the following classical elastic solutions (Xu and Liu 1995).

$$\begin{cases} \sigma_r = \frac{Pa^2}{a^2-b^2}(1 - \frac{b^2}{r^2}) \\ \sigma_\theta = \frac{Pa^2}{a^2-b^2}(1 + \frac{b^2}{r^2}) \end{cases} \tag{33}$$

$$u_r = -\frac{Pa^2(1+\nu)}{E(a^2 - b^2)}\left\{(1 - 2\nu)r + \frac{b^2}{r}\right\} \tag{34}$$

Effect of the tensile and compressive elastic parameters on the stress

In this section, a circular ring with an internal radius $a = 0.2\text{m}$ and an external radius $b = 0.5\text{m}$ is taken as an example; the distribution of stress, strain, and displacement along the radius is plotted. As illustrated in Fig. 5, the radial stress σ_r and the circumferential stress σ_θ both reduce with the increase of the radius r. When r reaches the outer radius $r = b$, σ_r tends to 0, but σ_θ is not zero. Secondly, it is found that the distribution pattern of σ_r and σ_θ along the radius is not influenced by the modulus ratio R, but the magnitudes of σ_r and σ_θ are significantly affected by the R. Besides, when the radius r is in the range $(0.285\text{m}, 0.500\text{m})$, there is a positive relationship between σ_r and R. However, if the radius r is in the range $(0.200\text{m}, 0.285\text{m})$, there is a negative relationship between σ_r and R. It is indicated that the circumferential stress near the inner wall is underestimated, and the circumferential stress near the outer wall is overestimated without considering the difference of tensile modulus and compressive modulus of materials.

Effect of the tensile and compressive elastic parameters on the strain

It is shown in Fig. 6 that ε_r and ε_θ are both reduced with the increasing of the radius r, and the bi-modulus characteristic of materials has a significant effect on the strain distribution in the circular ring. As illustrated in Fig. 6, when r is in the range of $(0.200\text{m}, 0.500\text{m})$, ε_θ increases significantly with the decrease of the parameter R, but the distribution and magnitude of ε_r are not affected by the increase of R. Summarily, the circumferential strain ε_θ and radial strain ε_r in the ring are underestimated without considering the difference between tensile modulus and compressive modulus of materials.

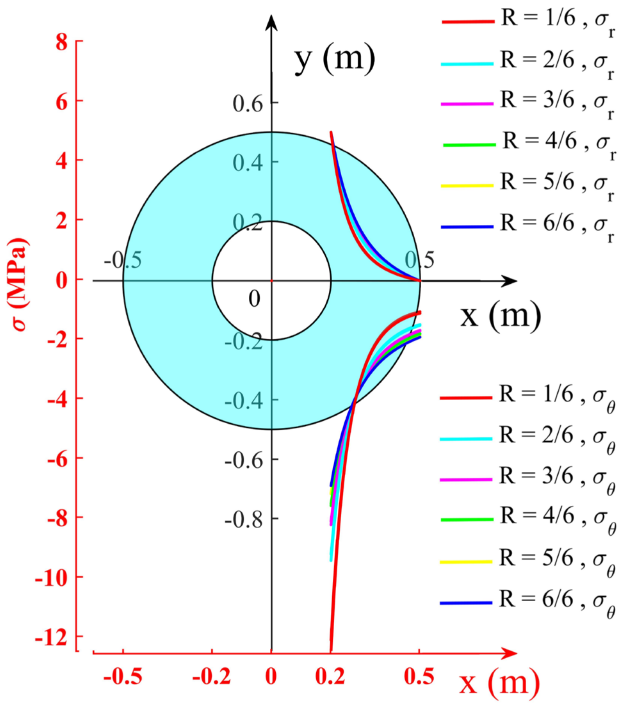


Fig. 5 Effect of the parameter R on the stress distribution in the ring ($R = E_t/E_c = v_t/v_c, P = 5\text{MPa}, E_c = 60\text{GPa}, v_c = 0.25, a = 0.2\text{m}, b = 0.5\text{m}$)

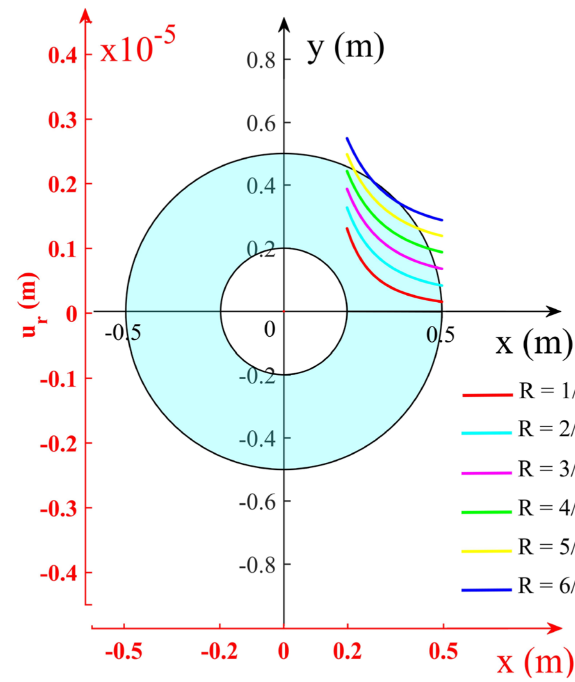


Fig. 7 Effect of the parameter R on the radial displacement distribution along the radius of the ring ($R = E_t/E_c = v_t/v_c, P = 5\text{MPa}, E_c = 60\text{GPa}, v_c = 0.25, a = 0.2\text{m}, b = 0.5\text{m}$)

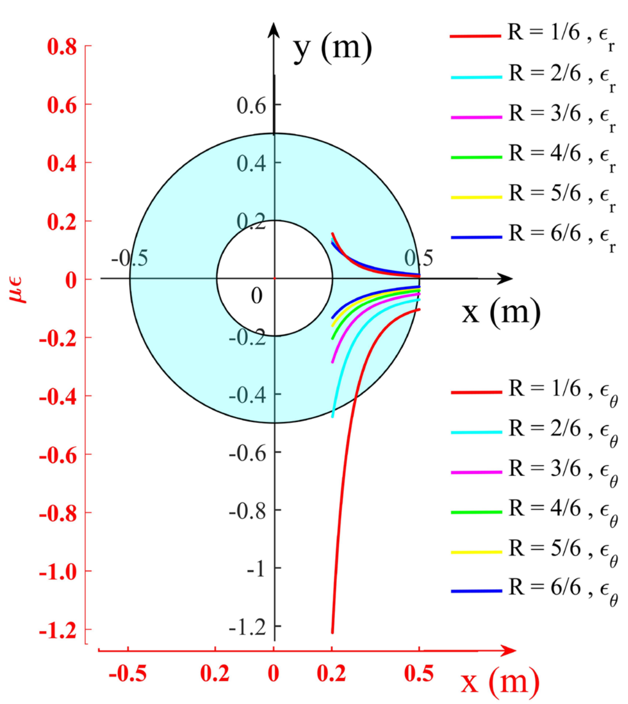


Fig. 6 Effect of the parameter R on the strain distribution in the ring ($R = E_t/E_c = v_t/v_c, P = 5\text{MPa}, E_c = 60\text{GPa}, v_c = 0.25, a = 0.2\text{m}, b = 0.5\text{m}$)

Effect of the tensile and compressive elastic parameters on the radius displacement

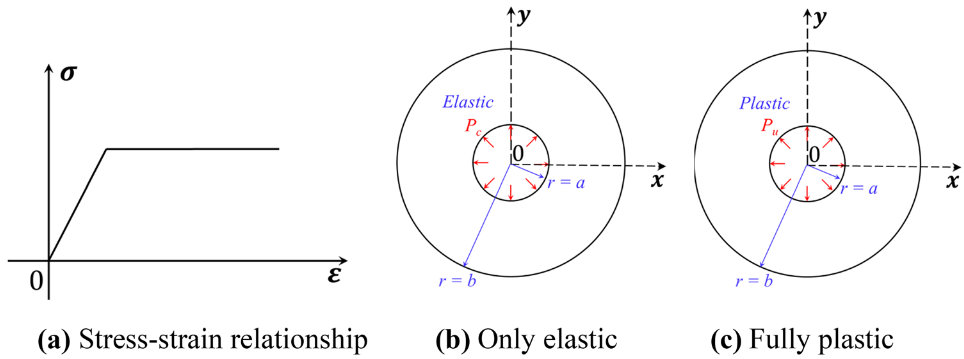
As can be seen in Fig. 7, the modulus ratio R has a significant influence on the radial displacement u_r. Summarily, it is very necessary to establish an analytical solution considering the bi-modulus characteristics of materials in the circular ring expansion problem.

Elastoplastic solution

If the material is an ideal elastoplastic material as shown in Fig. 8a, when the internal pressure on the inner wall satisfies $P = P_c$ (P_c is the critical internal pressure at which plastic deformation is just occurring on the inner wall), a plastic zone just appears at the internal radius i.e., $r = a$, as shown in Fig. 8b. It has been known that Eqs. (14) and (15) must be satisfied if the SMP yield criterion is used under plane strain condition. By substituting Eq. (30) into Eq. (15), the following expression is obtained:

$$\frac{Pa^{s+1}}{a^{2s} - b^{2s}}\left(r^{s-1} - \frac{b^{2s}}{r^{s+1}}\right) - K\frac{Pa^{s+1}}{a^{2s} - b^{2s}}\left(sr^{s-1} + \frac{sb^{2s}}{r^{s+1}}\right) = (K-1)\sigma_0 \tag{35}$$

Fig. 8 Diagram of the elasto-plastic solution for a circular ring applied by a uniform internal pressure for ideal elasto-plastic materials. **a** Stress-strain relationship. **b** Only elastic. **c** Fully plastic



By applying the following boundary condition, $P = P_c$ at $r = a$, the critical internal pressure P_c can be determined as follows:

$$P_c = \frac{(K-1)(b^{2s} - a^{2s})\sigma_0}{(Ks+1)b^{2s} + (Ks-1)a^{2s}} \tag{36}$$

When the internal pressure P on the inner wall $r = a$ increasing from P_c to the ultimate pressure P_u(under this pressure, the outer wall of the circular ring, i.e.,$r = b$, just enters the plastic state, as shown in Fig. 8c), the following boundary condition should be satisfied:

$$\begin{cases} \sigma_r = P_u, r = a \\ \sigma_r = P_c, r = b \end{cases} \tag{37}$$

Substituting the SMP yield criterion under plane strain condition, i.e., Eqs. (14) and (15) into Eq. (16), the differential equation of stress equilibrium for the circular ring expansion problem can be given as follows:

$$\frac{d\sigma_r}{dr} + \frac{(K-1)\sigma_r}{rK} + \frac{(K-1)\sigma_0}{rK} = 0 \tag{38}$$

The general solution of Eq. (38) for the radial stress is

$$\sigma_r = C' r^{-\left(\frac{K-1}{K}\right)} - \sigma_0 \tag{39}$$

where C' is an integral constant. By introducing the boundary condition Eq. (37) into Eq. (39), the expression of ultimate internal pressure P_u is obtained as follows:

$$P_u = \left(P_c + \sigma_0\right)\left(\frac{b}{a}\right)^{\frac{K-1}{K}} - \sigma_0 \tag{40}$$

As illustrated in Fig. 9, the modulus ratio R, i.e., $R = E_t/E_c = \nu_t/\nu_c$ and the friction angle φ of materials have a significant effect on the P_c and P_u. The critical internal pressure P_c and ultimate internal pressure P_u increase gradually with the increase of R. It is indicated that the critical internal pressure P_c and ultimate internal pressure P_u would be overestimated without considering the bi-modulus characteristics of materials.

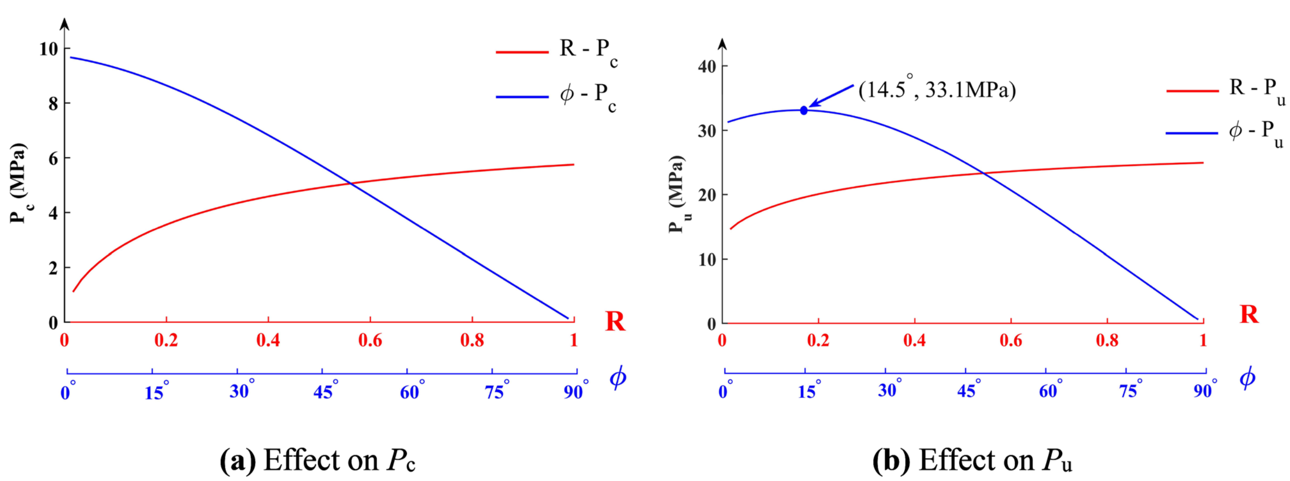


Fig. 9 Effect of the bi-modulus characteristic of materials on the critical internal pressure P_c and the ultimate pressure P_u of the circular ring ($(R = E_t/E_c = \nu_t/\nu_c)$. **a** Effect on P_c. **b** Effect on P_u

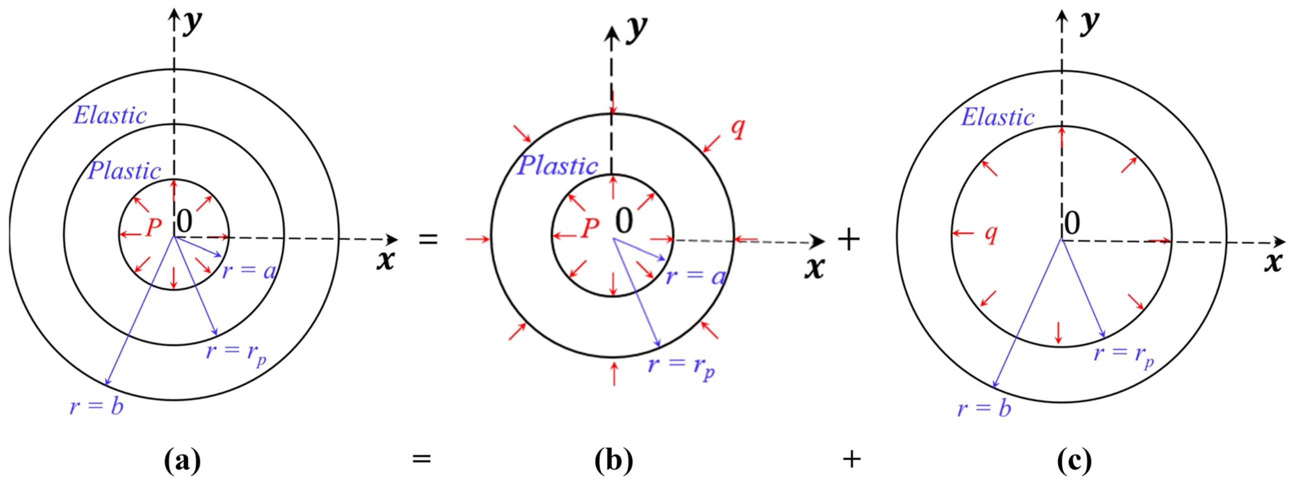


Fig. 10 Circular ring expansion with the coexistence of elastic and plastic zones. (**a**) = (**b**) + (**c**)

Elastroplastic solution of stress

It is known that the above critical internal pressure P_c and ultimate pressure P_u correspond to two very special cases. However, the most situation is that the elastic zone and the plastic zone coexist in the circular ring when $P_c < P < P_u$, as shown in Fig. 10a. As a result, the circular ring can be divided into two regions in this case, i.e., $a < r \le r_p$ in the circular ring is the elastic zone, as illustrated in Fig. 10b, and $r_p < r \le b$ is the plastic zone, as illustrated in Fig. 10c. On the interface $r = r_p$, the continuity of displacement and stress must be guaranteed. For the plastic zone, except the radially outward pressure P on the inner wall $r = a$, there is another radially inward pressure q applied on the interface $r = r_p$. Correspondingly, there is only a radially outward pressure q on the interface $r = r_p$ for the elastic zone.

When the internal pressure P on the inner wall $r = a$ of the circular ring satisfies $P_c < P < P_u$, there are both elastic and plastic zones in the circular ring. In this case, the inner wall of the ring is being in the plastic state and satisfies the following boundary condition:

$$\sigma_r = P_p, r = a \tag{41}$$

Substituting the above boundary condition into Eq. (39), the solution of radical stress σ_r can be obtained. Then, by further introducing the solution of σ_r into the SMP yield criterion, i.e., Eq. (15), the radical stress σ_r and circumferential stress σ_θ in the plastic zone can be obtained as follows:

$$\begin{cases} \sigma_r = \left(P + \sigma_0\right)\left(\frac{a}{r}\right)^{\frac{K-1}{K}} - \sigma_0 \\ \sigma_\theta = \frac{1}{K}\left(P + \sigma_0\right)\left(\frac{a}{r}\right)^{\frac{K-1}{K}} - \frac{K-1}{K}\sigma_0 \end{cases} \tag{42}$$

Meanwhile, substituting the boundary condition Eq. (41) into Eq. (42), the radical stress q on the interface $r = r_p$ can be expressed as follows:

$$q = \left(P + \sigma_0\right)\left(\frac{a}{r_p}\right)^{\frac{K-1}{K}} - \sigma_0 \tag{43}$$

Besides, for the outside elastic zone shown in Fig. 10c, the interface $r = r_p$ is about to enter plasticity. As a consequence, the following boundary condition is met at this moment:

$$\sigma_r = P_c, r = r_p \tag{44}$$

Under this situation, the expression of stress in the elastic zone can be given as follows:

$$\begin{cases} \sigma_r = \frac{P{r_p}^{s+1}}{{r_p}^{2s} - b^{2s}}\left(r^{s-1} - \frac{b^{2s}}{r^{s+1}}\right) \\ \sigma_\theta = \frac{P{r_p}^{s+1}}{{r_p}^{2s} - b^{2s}}\left(sr^{s-1} + \frac{sb^{2s}}{r^{s+1}}\right) \end{cases} \tag{45}$$

$$P_c = \frac{(K-1)(b^{2s} - {r_p}^{2s})\sigma_0}{(Ks+1)b^{2s} + (Ks-1){r_p}^{2s}} \tag{46}$$

Since the stress on the interface $r = r_p$ must be continuous, the radical stress P applied on the internal wall, i.e., $r = a$ can be formulated by substituting Eq. (43) into Eq. (46) as

$$P = \left[\sigma_0 + \frac{(K-1)(b^{2s} - {r_p}^{2s})\sigma_0}{(Ks+1)b^{2s} + (Ks-1){r_p}^{2s}}\right]\left(\frac{r_p}{a}\right)^{\frac{K-1}{K}} - \sigma_0 \tag{47}$$

In this section, the relationship between the internal pressure P and the radius r_p of the elastoplastic interface

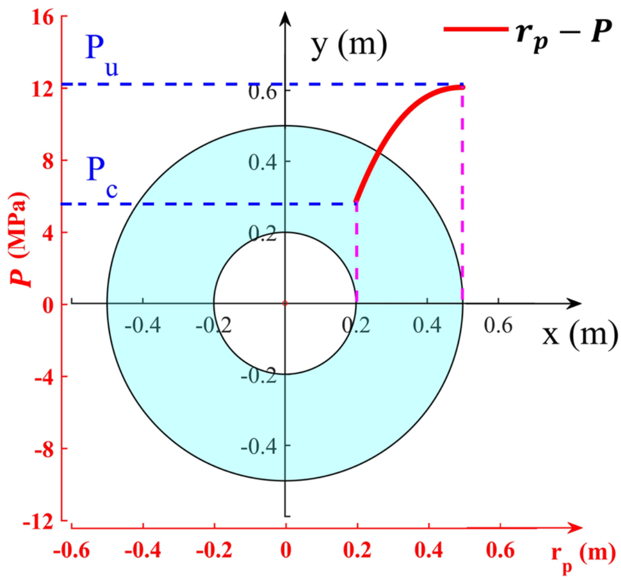


Fig. 11 Effect of the internal pressure P on the radius r_p of the elasto-plastic interface

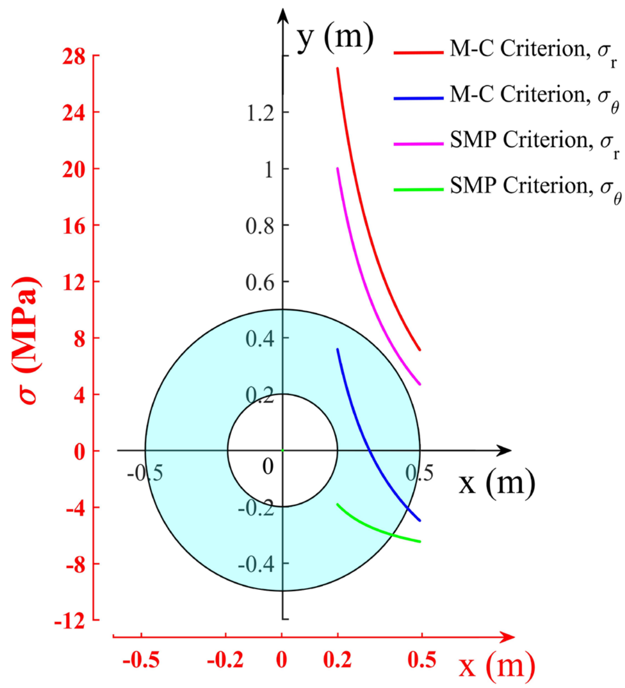


Fig. 12 Stress comparison between the SMP yield criterion and the M-C yield criterion for the axisymmetric circular ring expansion problem under plane strain condition

is explored, the material parameters are set as $C = 10\text{MPa}$, $E_c = 60\text{GPa}$, $E_t = 60\text{GPa}, v_c = 0.25, v_t = 0.25, \varphi = 45°$. According to the $r_p - P$ relationship, as illustrated in Fig. 11, there is a positive relationship between r_p and P. It is observed that the relationship is nearly linear when P is close to P_c, and it is nonlinear when P close to P_u.

Elastoplastic solution of radial displacement

As shown in Fig. 10, the circular ring has been divided into two zones (i.e., $a < r \le r_p$ is the plastic zone and $r_p < r \le b$ is the elastic zone). In this case, the following boundary condition should be satisfied on the elastoplastic interface $r = r_p$:

$$\begin{cases} \sigma_r = q, r = r_p \\ q = P_c \end{cases} \tag{48}$$

Substituting Eq. (48) into Eq. (32), the radical displacement in the elastic zone can be obtained as

$$u_r = -\frac{P_c {r_p}^{s+1}}{E_c({r_p}^{2s} - b^{2s})}\left\{\left[\left(1 - v_c v_c\right) - s(v_c + v_c v_c)\right]\frac{r^s}{s} + \left[\left(1 - v_c v_c\right) + s(v_c + v_c v_c)\right]\frac{b^{2s}}{s r^s}\right\} \tag{49}$$

Due to the fact that the relationship between stress and strain in the plastic zone can't be analytically given, because there is no one-to-one corresponding relationship between stress and strain at yielding state for ideal elastoplastic materials, it is difficult to give the analytical solution of the radical displacement in the plastic zone in the circular ring. The solution of the radical displacement could usually be determined by the numerical methods.

Comparison of stress in the plastic zone between M-C criterion and SMP criterion

SMP yield criterion was evolved from the Mohr-Coulomb yield criterion. Compared with the Mohr-Coulomb yield criterion, the SMP yield criterion considers the effect of the intermediate principal stress σ_2. To explore the difference of stress between the two yield criteria for the circular ring expansion problem of ideal elastoplastic materials, the comparison between the results determined by Eq. (42) in this study and the following Eqs. (50) and (51) proposed based on the Mohr-Coulomb yield criterion Luo et al. (2010) is performed. The comparison results are demonstrated in Fig. 12.

$$\begin{cases} P_c = \frac{2C\cos\varphi}{1+a-(1-a)sin\varphi} \\ P_u = Ccot\varphi\left[\frac{(1+a)(1+\sin\varphi)}{1+a-(1-a)\sin\varphi}\left(\frac{b}{a}\right)^{\frac{2\sin\varphi}{1+\sin\varphi}} - 1\right] \\ a = \sqrt{\frac{E_t(1-\nu_c\nu_c)}{E_c(1-\nu_c\nu_t)}} \end{cases} \tag{50}$$

$$\begin{cases} \sigma_r = \left(P_p + C\cot\varphi\right)\left(\frac{a}{r}\right)^{\frac{2\sin\varphi}{1+\sin\varphi}} - Ccot\varphi \\ \sigma_\theta = \frac{1-\sin\varphi}{1+\sin\varphi}\left(P_p + C\cot\varphi\right)\left(\frac{a}{r}\right)^{\frac{2\sin\varphi}{1+\sin\varphi}} - Ccot\varphi \end{cases} \tag{51}$$

The material parameters are set as $C = 10\text{MPa}, E_c = 60\text{GPa}$, $E_t = 60\text{GPa}, \nu_c = 0.25, \nu_t = 0.25, \varphi = 40°$. $P_c = 6.3\text{MPa}$, and $P_u = 27.0\text{MPa}$ are determined according to Eqs. (36) and (40) which are derived based on the SMP yield criterion. According to Eq. (50), the critical internal pressure P_c and the ultimate internal pressure P_u under the M-C yield criterion are 7.7MPa and 28.1MPa, respectively. To ensure the circular ring can get into a plastic state under the two yield criteria. The internal pressure was set on the inner wall of the circular ring, $P = 28.5\text{MPa}$. Here, the radial stress σ_r and the circumferential stress σ_θ determined by the SMP yield criterion and M-C yield criterion are compared.

Figure 12 shows the comparison of the distribution pattern and the magnitude of σ_r and σ_θ along the radius determined by the SMP yield criterion and Mohr-Coulomb yield criterion. It is indicated that the magnitude of σ_r determined by Mohr-Coulomb yield criterion is slightly greater than that determined by the SMP yield criterion, but their distribution pattern is basically the same. However, the difference on σ_θ between the two yield criteria is significant. The main cause for this difference is that the effect of the intermediate principal stress σ_2 is considered in the SMP yield criterion. Therefore, it is recognized that the stress in the plastic zone will be significantly overestimated without considering the influence of the intermediate principal stress.

Conclusion

1. In this study, the generalized analytical solutions for the axisymmetric circular ring expansion problem under plane strain condition are proposed based on the SMP yield criterion and bi-modulus theory. Most importantly, the difference between the elastic parameters of materials under tension and compression has been successfully considered in these solutions.
2. Compared with the Mohr-Coulomb yield criterion, the effect of the intermediate principal stress has been considered in the SMP yield criterion. As a result, the analytical solutions proposed in this study based on the SMP yield criterion could better describe the stress state in the plastic zone of bi-modulus materials for the circular ring expansion problem.
3. The effects of the modulus ratio R, i.e., $R = E_t/E_c = \nu_t/\nu_c$, on the stress, strain, and displacement, respectively, have been investigated by considering the bi-modulus characteristic of materials. It is known, by the comparative analysis that the difference of elastic parameters under tension and compression, i.e.,E_t,E_c, ν_t, ν_c, has a significant effect on the distribution pattern and magnitude of the stress σ_r, σ_θ, the strain ε_r, ε_θ, the displacement u_r and the magnitude of critical internal pressure P_c, and the ultimate internal pressure P_u. Therefore, the classical analytical solution for the circular ring expansion problem without considering the difference of material parameters under tension and compression is defective, and the work presented in this study could make up for this defect, to some extent.

Author contribution Conceptualization, Jianhong Ye and Tianming Wang. Formal analysis, Tianming Wang and Jianhong Ye. Funding acquisition, Jianhong Ye. Investigation, Tianming Wang and Jianhong Ye. Methodology, Tianming Wang and Jianhong Ye. Project administration, Jianhong Ye. Supervision, Jianhong Ye. Visualization, Tianming Wang. Writing–original draft, Tianming Wang. Writing–review and editing, Jianhong Ye

Funding This work is grateful for the funding support from the National Natural Science Foundation of China, Grant No. 51879257.

Data availability Upon request.

Declarations

Conflict of interest The authors declare no competing interests.

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